

UPPER BOUND FOR THE GROMOV WIDTH OF COADJOINT ORBITS OF TYPE A

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ABSTRACT. We find an upper bound for the Gromov width of coadjoint orbits of $U(n)$ with respect to the Kirillov-Kostant-Souriau symplectic form by computing certain Gromov-Witten invariants. The approach presented here is closely related to the one used by Gromov in his celebrated non-squeezing theorem.

This is a preliminary version. Comments are welcome.

1. INTRODUCTION

The Darboux theorem in symplectic geometry states that around any point of a symplectic manifold, there is a system of local coordinates such that the symplectic manifold looks locally like \mathbb{C}^n with its canonical symplectic form. A natural and fundamental problem in symplectic geometry is to know how far we can extend symplectically these coordinates in the symplectic manifold. This is how the concept of Gromov's width arises. The Gromov width of a symplectic manifold (M, ω) is defined as

$$\text{Gwidth}(M, \omega) = \sup \{ \pi r^2 : \exists \text{ a symplectic embedding } B_{2n}(r) \hookrightarrow M \}.$$

Roughly speaking, the Gromov width of a symplectic manifold is a measure of its symplectic size. Gromov's width was first introduced by Gromov in [9] and it has lead to the notion of symplectic capacities [5].

It is interesting to know how big or small can be the Gromov width. For example, it has been conjectured by Paul Biran that if the cohomology class of the symplectic form of a symplectic manifold is integral, then the Gromov width of the symplectic manifold is at least one. On the other hand, the Gromov non-squeezing theorem gives us insights of how restrictive is the Gromov width from above:

Gromov's non-squeezing Theorem *If ρ is a symplectic embedding of the ball $B_{2n}(r)$ of radius r into a cylinder $B^2(\lambda) \times \mathbb{R}^{2n-2}$ of radius λ , then $r \leq \lambda$. In particular,*

$$\text{Gwidth}(B^2(\lambda) \times \mathbb{R}^{2n-2}) = \pi \lambda^2.$$

Gromov's non-squeezing Theorem is frequently considered as a classical mechanics counterpart of the Heisenberg's Uncertainty Principle [4].

Gromov proved the non-squeezing theorem in [9], where he established the connection between J -holomorphic curves and symplectic geometry. Since then, several authors have used Gromov's

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method for bounding the Gromov width of other families of symplectic manifolds, such as G. Lu for symplectic toric manifolds in [18], Yael Karshon and Susan Tolman for complex Grassmannians manifolds in [15] and Masrour Zoghi for regular coadjoint orbits in [25] (see also McDuff-Polterovich [19], Biran [2]).

In this paper, we are particularly interested in finding upper bounds for the Gromov width of general coadjoint orbits of $U(n)$. We identify the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ with its dual $\mathfrak{u}(n)^*$ via the invariant inner product defined by the formula

$$(X, Y) = \text{Trace } XY.$$

The mapping $h \mapsto ih$ is an isomorphism from the real vector space of Hermitian matrices $\mathcal{H} := i\mathfrak{u}(n)$ onto the real vector space of skew-Hermitian matrices $\mathfrak{u}(n)$. This isomorphism together with the invariant inner product allow us to identify \mathcal{H} with $\mathfrak{u}(n)^*$, and a set of Hermitian matrices that share the same spectrum with a coadjoint orbit of $U(n)$, i.e., for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ there exists a coadjoint orbit of $U(n)$ which can be identified with $\mathcal{H}_\lambda := \{A \in M_n(\mathbb{C}) : A^* = A, \text{spectrum } A = \lambda\}$. In this case, we can endow \mathcal{H}_λ with a symplectic form ω_λ coming from the Kostant-Kirillov-Souriau symplectic form defined on the coadjoint orbit of $U(n)$.

The main result obtained in this paper is that if there are i, j such that any difference of eigenvalues $\lambda_{i'} - \lambda_{j'}$ is an integer multiple of $\lambda_i - \lambda_j$, then

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq |\lambda_i - \lambda_j|.$$

This result is an extension of one that Masrour Zoghi has obtained in his Ph.D thesis [25], where he has considered the problem of determining the Gromov width of *regular coadjoint orbits* of compact Lie groups. Recall that a coadjoint orbit of a compact Lie group is regular if the stabilizer of any element of it under the coadjoint action is a maximal torus of the compact Lie group. When the compact Lie group is the group of unitary matrices $U(n)$, a coadjoint orbit is regular if and only if it can be identified with a set of the form \mathcal{H}_λ with all the components of $\lambda \in \mathbb{R}^n$ being pairwise different. Our results are extended to coadjoint orbits of $U(n)$ that are not necessarily regular.

We expect to obtain a similar result for *any* coadjoint orbit of *any* simple compact Lie group, but this would be described in a later paper.

This paper is organized as follows: we first introduce the necessary J -holomorphic tools that we will use throughout the text, and we then explain how upper bounds for the Gromov width of symplectic manifolds can be given by a non-vanishing Gromov-Witten invariant.

Then we show how upper bounds for the Gromov width of Grassmannian manifolds can be found by computing certain Gromov-Witten invariant. The problem of finding the Gromov width for Grassmannians manifolds has been already considered and solved independently by Yael Karshon and Susan Tolman in [15] and by Guangcun Lu in [17]. The ideas presented in this paper are similar in nature to the ones used by Karshon and Tolman in their paper.

Finally, we show how these considerations about Grassmannian manifolds would be particularly useful for working out the most general problem of determining upper bounds for the Gromov width of partial flag manifolds. The reason of this is that in some particular cases computations of

Gromov-Witten invariants for partial flag manifolds can be reduced to computations of Gromov-Witten invariants for Grassmannians manifolds.

We suggest to the reader to compare our results with the results obtained by Milena Pabiniak in [21], where she considers the problem of determining *lower bounds* for the Gromov width of coadjoint orbits of $U(n)$ by using equivariant techniques of symplectic geometry. In her paper, Pabiniak proves that for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ of the form

$$(1) \quad \lambda_1 > \lambda_2 > \dots > \lambda_l = \lambda_{l+1} = \dots = \lambda_{l+s} > \lambda_{l+s+1} > \dots > \lambda_n; s \geq 0,$$

the Gromov width of $(\mathcal{H}_\lambda, \omega_\lambda)$ is at least the minimum $\min\{\lambda_i - \lambda_j : \lambda_i > \lambda_j\}$. This result together with the one obtained in this paper, implies that if $\lambda \in \mathbb{R}^n$ is of the form (1) and if there are i, j such that any difference of the form $\lambda_{i'} - \lambda_{j'}$ is an integer multiple of $\lambda_i - \lambda_j$, then

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) = |\lambda_i - \lambda_j|;$$

suggesting that the upper bound that we have found is indeed the Gromov width of $(\mathcal{H}_\lambda, \omega_\lambda)$.

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2. J -HOLOMORPHIC CURVES

Pseudoholomorphic theory has been one of the main tools used in symplectic geometry since Gromov introduced them in [9] where he proved his celebrated non-squeezing theorem. We want to apply similar ideas for finding upper bounds for the Gromov width of coadjoint orbits of type A, or partial flag manifolds. In this section we give a short review of pseudoholomorphic theory and Gromov-Witten invariants, and we show how they are related with the Gromov width of a symplectic manifold.

2.1. Pseudoholomorphic theory. Let (M^{2n}, ω) be a symplectic manifold. An almost complex structure J of (M, ω) is a smooth operator $J : TM \rightarrow TM$ such that $J^2 = -Id$. We say that an almost complex structure J is **compatible** with ω if the formula

$$g(v, w) := \omega(v, Jw)$$

defines a Riemannian metric. We denote the space of ω -compatible almost complex structures by $\mathcal{J}(M, \omega)$.

Let (\mathbb{CP}^1, j) be the Riemann sphere with its standard complex structure j . Let $J \in \mathcal{J}(M, \omega)$. A map $u : \mathbb{CP}^1 \rightarrow M$ is called a **J -holomorphic curve of genus zero** or simply a **J -holomorphic curve** if

$$J \circ du = du \circ j.$$

The nonlinear Cauchy Riemman operator $\bar{\partial}$ is defined using the formula

$$\begin{aligned} \bar{\partial}_J : C^\infty(\mathbb{CP}^1, M) &\rightarrow \bigcup_{u \in C^\infty(\mathbb{CP}^1, M)} \Omega^{0,1}(\mathbb{CP}^1, u^*TM) \\ u &\mapsto \frac{1}{2}(du + J \circ du \circ j) \end{aligned}$$

where the codomain is considered as a bundle over $C^\infty(\mathbb{CP}^1, M)$, $\bar{\partial}_J$ is considered as a section of this bundle, $u \in C^\infty(\mathbb{CP}^1, M)$ and $u^*TM = \{(z, v) : z \in \mathbb{CP}^1, v \in T_{u(z)}M\}$.

A curve $u : \mathbb{CP}^1 \rightarrow M$ is said to be **multiply covered** if it is the composite of a holomorphic branched covering map $(\mathbb{CP}^1, j) \rightarrow (\mathbb{CP}^1, j)$ of degree greater than one with a J -holomorphic map $\mathbb{CP}^1 \rightarrow M$. It is **simple** if it is not multiply covered.

Given a compact symplectic manifold (M^{2n}, ω) , a compatible almost complex structure J , and a second homology class $A \in H_2(M, \mathbb{Z})$, we define the **moduli space of simple J -holomorphic curves of degree A** as

$$\mathcal{M}_A^*(M, J) = \{u : \mathbb{CP}^1 \rightarrow M : J \circ du = du \circ j, u_*[\mathbb{CP}^1] = A, u \text{ is simple}\}.$$

The almost complex structure J is called **regular for A** if for every $u \in \mathcal{M}_A^*(M, J)$ such that $\bar{\partial}_J u = 0$, the vertical differential of the nonlinear Cauchy-Riemann operator $\bar{\partial}_J$ at the point u is surjective onto $\Omega^{0,1}(\mathbb{CP}^1, u^*TM)$. If an almost complex structure J is regular for every $A \in H_2(M, \mathbb{Z})$, then it will simply be called **regular**. The set of regular ω -compatible almost complex structures is residual in the set $\mathcal{J}(M, \omega)$ of compatible almost complex structures, i.e., it contains a countable intersection of open dense sets with respect to the C^∞ topology.

If J is a regular almost complex structure, then the moduli space $\mathcal{M}_A^*(M, J)$ is a smooth oriented manifold of dimension equal to $\dim M + 2c_1(A)$, where c_1 denotes the first Chern class of the bundle (TM, J) [20].

Example 2.1. If (M, ω, J) is a compact Kähler manifold and G is a Lie group such that acts transitively on M by holomorphic diffeomorphism, then the almost complex structure J is regular [20, Proposition 7.4.3].

A homology class $B \in H_2(M)$ is **spherical** if it is in the image of the Hurewicz homomorphism $\pi_2(M) \rightarrow H_2(M)$. A homology class $B \in H_2(M)$ is **ω -indecomposable** if it does not decompose as a sum $B = B_1 + \dots + B_k$ of spherical classes such that $\omega(B_i) > 0$. Gromov's compactness theorem [20] implies that when $A \in H_2(M, \mathbb{Z})$ is a ω -indecomposable homology class and J is a regular almost complex structure, the moduli space $\mathcal{M}_A(M, J)/PSL(2, \mathbb{C})$ of unparametrized J -holomorphic curves of degree A is compact.

In general, moduli spaces of pseudoholomorphic curves are not compact but can be compactified by adding sets of stable maps [20].

The **moduli space of simple J -holomorphic curves of degree A with k -marked points** is defined by

$$\mathcal{M}_{A,k}^*(M, J) = \mathcal{M}_A^*(M, J) \times_{PSL(2, \mathbb{C})} (\mathbb{CP}^1)^k$$

where $PSL(2, \mathbb{C})$ acts on the right factor by its natural action on \mathbb{CP}^1 and on the left factor by reparametrization. When $k = 0$, we define $\mathcal{M}_{A,0}^*(M, J)$ as being equal to $\mathcal{M}_A(M, J)/PSL(2, \mathbb{C})$.

We also have an **evaluation map**

$$\text{ev}_J^k := \mathcal{M}_{0,k}^*(M, A, J) = \mathcal{M}^*(M, A, J) \times_{PSL(2, \mathbb{C})} (\mathbb{CP}^1)^k \rightarrow M^k$$

defined by

$$\text{ev}_J^k[u, z_1, \dots, z_k] = (u(z_1), \dots, u(z_k)).$$

A smooth homotopy of almost complex structures is a smooth family $t \mapsto J_t, t \in [0, 1]$. For any such homotopy define

$$\mathcal{M}_{A,k}^*(M, \{J_t\}_t) = \{(t, u) : u \in \mathcal{M}_{A,k}^*(M, J_t)\}.$$

Given two regular ω -compatible almost complex structures J_0, J_1 we always can find a smooth homotopy of almost complex structures $\{J_t\}_t$ connecting them such that the space $\mathcal{M}_{A,k}^*(M, \{J_t\}_t)$ is a smooth oriented manifold of dimension $\dim M + 2c_1(A) + 2k - 5$ with boundary $\mathcal{M}_{A,k}^*(M, J_1) \sqcup \mathcal{M}_{A,k}^*(M, J_0)$, and with a smooth evaluation map

$$\text{ev}_{J_t}^k : \mathcal{M}_{A,k}^*(M, \{J_t\}_t) \rightarrow M$$

such that

$$\text{ev}_{J_t}^k|_{\partial \mathcal{M}_{A,k}^*(M, \{J_t\}_t)} = \text{ev}_{J_0}^k \sqcup \text{ev}_{J_1}^k : \mathcal{M}_{A,k}^*(M, J_1) - \mathcal{M}_{A,k}^*(M, J_0) \rightarrow M.$$

2.2. Gromov's width.

Definition 2.2. *Given a symplectic manifold (M^{2n}, ω) , its Gromov's width is defined as*

$$\text{Gwidth}(M, \omega) = \sup \{ \pi r^2 : \exists \text{ a symplectic embedding } B_{2n}(r) \hookrightarrow M \}.$$

The Darboux theorem implies that the Gromov width of a symplectic manifold is always positive. Moreover, if the symplectic manifold is compact, its Gromov's width is finite.

Theorem 2.3. *Let (M^{2n}, ω) be a compact symplectic manifold, and $A \in H_2(M, \mathbb{Z}) \setminus \{0\}$ a second homology class. Suppose that for a dense subset of smooth ω -compatible almost complex structures, the evaluation map*

$$\text{ev}_J^1 : \mathcal{M}_{A,1}^*(M, J) \rightarrow M$$

is onto. Then for any symplectic embedding $B_{2n}(r) \hookrightarrow M$, we have

$$\pi r^2 \leq \omega(A),$$

where $\omega(A)$ denotes the symplectic area of A . In particular,

$$\text{Gwidth}(M, \omega) \leq \omega(A).$$

Proof. Suppose that there is symplectic embedding

$$\rho : B_{2n}(r) \hookrightarrow M.$$

Fix an $\epsilon \in (0, r)$, let \tilde{J} be an ω -compatible complex structure on M that equals $\rho_*(J_{st})$ on the open subset $\rho(B_{2n}(r - \epsilon)) \subset M$.

We claim that there exists a \tilde{J} -holomorphic curve $\tilde{u} \in \mathcal{M}_B^*(M, \tilde{J})$ and $z \in \mathbb{CP}^1$ with $\text{ev}_{\tilde{J}}^1[\tilde{u}, z] = \tilde{u}(z) = \rho(0)$, where $0 \in B_{2n}(r - \epsilon)$ is the centre of the ball and $B \in H_2(M)$ satisfies $\omega(B) \leq \omega(A)$: If \tilde{J} is one of the almost complex structures for which $\text{ev}_{\tilde{J}}^1$ is onto,

then we are done. Otherwise, consider a sequence of ω -compatible almost complex structures $\{J_k\}_{k=1}^\infty$ that C^∞ -converge to \tilde{J} and for which $\text{ev}_{J_k}^1$ is onto and choose $u_k : \mathbb{CP}^1 \rightarrow M$ such that $\rho(0) \in u_k(\mathbb{CP}^1)$. By Gromov's compactness, the sequence $\{(u_k, J_k)\}$ has a subsequence $\{(u'_l, J'_l)\}_{l=1}^\infty \subset \{(u_k, J_k)\}$ Gromov converging to a stable map

$$u^s : \mathbb{CP}^1 \sqcup \dots \sqcup \mathbb{CP}^1 \rightarrow M$$

whose image contains $\rho(0)$. Now, let $\tilde{u} : \mathbb{CP}^1 \rightarrow M$ be the restriction of u^s to the component of the domain of u^s that contains the marked point. Moreover, let $B = \tilde{u}_*([\mathbb{CP}^1])$, then it satisfies

$$\omega(B) \leq \omega(A).$$

Since \tilde{u} is \tilde{J} -holomorphic, its restriction to $S := \tilde{u}^{-1}(\rho(B_{2n}(r - \epsilon))) \subset \mathbb{CP}^1$ gives a proper holomorphic curve $u' : S \rightarrow B_{2n}(r - \epsilon)$ that passes through the origin. By an standard fact in minimal surface theory, the area of this holomorphic curve is bounded from below by $\pi(r - \epsilon)^2$, whereas $\text{area}(u') \leq \text{area}(\tilde{u}) = \omega(B) \leq \omega(A)$, and so $\pi(r - \epsilon)^2 \leq \omega(A)$. Since this equality is true for all $\epsilon > 0$, we conclude that

$$\pi r^2 \leq \omega(A).$$

□

In order to find upper bounds for the Gromov width of a symplectic manifold (M, ω) , we want to prove that for generic ω -compatible almost complex structures J , the evaluation map

$$\text{ev}_J^1 : \mathcal{M}_{A,1}^*(M, J) \rightarrow M$$

is onto. One way to achieve the ontoeness of the evaluation map is for example by proving that a Gromov-Witten invariant with one of its constraints being a point is different from zero.

Gromov-Witten invariants are well defined, at least if we assume that either the symplectic manifold (M, ω) is semipositive or the the homology class $A \in H_2(M; \mathbb{Z})$ is ω -indecomposable, a symplectic manifold (M, ω) is **semipositive** if, for a spherical homology class A with positive symplectic area, $c_1(A) \geq 3 - n$ implies $c_1(A) \geq 0$. In these cases, for a regular almost complex structure J of (M, ω) , the evaluation map

$$\text{ev}_J^k : \mathcal{M}_{A,k}^*(M, J) \rightarrow M^k$$

represents a pseudocycle, i.e., its image can be compactified by adding a set of codimension at least two.

If $a_i \in H^*(M)$ are cohomology classes Poincaré dual to compact oriented submanifolds $X_i \subset M$, the **Gromov-Witten invariant** $\text{GW}_{A,k}^J(a_1 \cdots a_k)$ is the number of J -holomorphic spheres in the class A passing through the submanifolds X_i (after possibly perturbing them) and counted with appropriate signs. More precisely, if $\sum_{i=1}^k \deg a_i = \dim \mathcal{M}_{A,k}^*(M, J)$ and the moduli space $\mathcal{M}_{A,k}^*(M, J)$ is endowed with a suitable orientation (see, e.g., [20, Section A.2]); the Gromov-Witten invariant is defined as the intersection oriented number

$$\text{GW}_{A,k}^J(a_1 \cdots a_k) := \sharp \text{ev}_J^k \cap (X_1 \times \cdots \times X_k).$$

If we do not orient the moduli space $\mathcal{M}_{A,k}^*(M, J)$, we can still define Gromov-Witten invariants over \mathbb{Z}_2 . Gromov-Witten invariants $\text{GW}_{A,k}^J$ are well-defined, finite and independent of the regular almost complex structure J [20, Theorem 7.1.1, Lemma 7.1.8].

Remark 2.4. Note that if there exist cohomology classes a_1, \dots, a_k and a suitable regular almost complex structure J such that $\text{GW}_{A,k}^J(a_1 \cdots a_k) \neq 0$ and a_1 is Poincaré dual to the fundamental class of a point, then for a generic choice of almost complex structure J' , the evaluation map

$$\text{ev}_{J'}^1 : \mathcal{M}_{A,1}^*(M, J') \rightarrow M$$

is onto, which, by Theorem 2.3, implies that

$$\text{Gwidth}(M, \omega) \leq \omega(A).$$

Remark 2.5. Gromov-Witten invariants for symplectic manifolds can be defined in wide generality by associating to the moduli spaces of J -holomorphic curves *virtual fundamental classes* with rational coefficients (Li-Tian [16], Fukaya-Ono [6], Ruan [22], Siebert [23], Hofer-Wysocki-Zehnder [13], [12]). We will not make use of this definition since we want to keep as simple and self-contained as possible the presentation of this paper. However, with this definition we would not need to assume that either the symplectic manifold is semipositive or the homology class A is indecomposable, and the results of Theorem 5.4 can be extended to *any* coadjoint orbit of type A.

3. COADJOINT ORBITS OF TYPE A

The coadjoint orbits of a compact Lie group are endowed with a symplectic form known as the Kostant-Kirillov-Souriau form. We wish to apply to this family of symplectic manifolds, pseudoholomorphic tools for studying the Gromov width. We focus our attention in coadjoint orbits of type A, or partial flag manifolds. In this section we recall some general statements about coadjoint orbits.

Let G be a compact Lie group, \mathfrak{g} be its Lie algebra, and \mathfrak{g}^* be the dual of the Lie algebra \mathfrak{g} . The compact Lie group G acts on \mathfrak{g}^* by the coadjoint action. Let $\xi \in \mathfrak{g}^*$ and \mathcal{O}_ξ be the coadjoint orbit through ξ .

The coadjoint orbit \mathcal{O}_ξ carries a symplectic form defined as follows: for $\xi \in \mathfrak{g}^*$ we define a skew bilinear form on \mathfrak{g} by

$$\omega_\xi^{KKS}(X, Y) = \langle \xi, [X, Y] \rangle.$$

The kernel of ω_ξ^{KKS} is the Lie algebra \mathfrak{g}_ξ of the stabilizer of $\xi \in \mathfrak{g}^*$ for the coadjoint representation. In particular, ω_ξ^{KKS} defines a nondegenerate skew-symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_\xi$, a vector space that can be identified with $T_\xi(\mathcal{O}_\xi) \subset \mathfrak{g}^*$. The bilinear form ω_ξ^{KKS} induces a closed, invariant, nondegenerate 2-form on the orbit \mathcal{O}_ξ , therefore defining a symplectic structure on \mathcal{O}_ξ . This symplectic form is known as the **Kostant-Kirillov-Souriau form** of the coadjoint orbit.

Let us assume now that $G = U(n)$. Let $\mathfrak{u}(n)$ be the Lie algebra of $U(n)$, $\mathfrak{u}(n)^*$ be its dual and $\mathcal{H} = \{A \in M_n(\mathbb{C}) : A^* = A\}$ be the set of Hermitian matrices.

The group of unitary matrices $U(n)$ acts by conjugation on \mathcal{H} . The Hermitian matrices \mathcal{H} have real eigenvalues and are diagonalizable in a unitary basis, so that the orbits of this action correspond to sets of matrices in \mathcal{H} with the same spectrum. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $\mathcal{H}_\lambda = \{A \in M_n(\mathbb{C}) : A^* = A, \text{ spectrum } A = \lambda\}$ be the $U(n)$ -orbit of the matrix $\text{diagonal}(\lambda_1, \dots, \lambda_n)$ in \mathcal{H} .

We identify $U(n)$ -orbits in \mathcal{H} with adjoint orbits in $\mathfrak{u}(n)$ by sending a matrix $A \in \mathcal{H}$ to the matrix $iA \in \mathfrak{u}(n)$. The pairing in $\mathfrak{u}(n) = i\mathcal{H}$ defined by

$$(X, Y) = \text{Trace}(XY)$$

allows us to identify $\mathfrak{u}(n)$ with $\mathfrak{u}(n)^*$, and adjoint orbits in $\mathfrak{u}(n)$ with coadjoint orbits in $\mathfrak{u}(n)^*$. So that, $U(n)$ -orbits in \mathcal{H} can be identified with coadjoint orbits in $\mathfrak{u}(n)^*$.

Under these identifications, for $\lambda \in \mathbb{R}^n$, \mathcal{H}_λ can be identified with a coadjoint orbit in $\mathfrak{u}(n)^*$. In this case, we define a symplectic form ω_λ on \mathcal{H}_λ by pulling back the Kirillov-Kostant-Souriau form defined on the coadjoint orbit. We also endow \mathcal{H}_λ with a complex structure J_λ , coming from the presentation of \mathcal{H}_λ as a quotient of complex Lie groups $Sl(n, \mathbb{C})/P$, where $P \subset Sl(n, \mathbb{C})$ is a parabolic subgroup of block upper triangular matrices. The triple $(\mathcal{H}_\lambda, \omega_\lambda, J_\lambda)$ is a Kähler manifold and the Lie group $Sl(n, \mathbb{C})$ acts holomorphically and transitively on \mathcal{H}_λ by conjugation.

Let $\{e_i\}_{i=1}^n$ denote the standard basis of \mathbb{R}^n . Let $T = U(1)^n \subset U(n)$ be the standard maximal torus of $U(n)$ and $\mathfrak{t} \cong \mathbb{R}^n$ be its Lie algebra. We identify \mathfrak{t}^* with \mathfrak{t} via its standard inner product so that the standard basis $\{e_i\}_{i=1}^n$ of $\mathfrak{t} \cong \mathbb{R}^n$ is identified with the standard basis of projections of \mathfrak{t}^* , which is also the standard basis (as a \mathbb{Z} -module) of the weight lattice $\text{Hom}(T, S^1) \subset \mathfrak{t}^*$.

The restricted action of $T \subset U(n)$ on \mathcal{H}_λ is Hamiltonian with momentum map

$$\begin{aligned} \mu : \mathcal{H}_\lambda &\rightarrow \mathfrak{t}^* \simeq \mathbb{R}^n \\ (a_{ij}) &\mapsto (a_{11}, \dots, a_{nn}). \end{aligned}$$

The image of the momentum map is the convex hull of the momentum images of the fixed points of the action of T on \mathcal{H}_λ , i.e., the image of μ is the convex hull of all possible permutations of the vector $(\lambda_1, \dots, \lambda_n)$ (see, e.g., [1, Chapter III], [11]).

The $U(n)$ -orbit \mathcal{H}_λ together with the torus T action is a **GKM space**, i.e., the closure of every connected component of the set $\{x \in \mathcal{H}_\lambda : \dim_{\mathbb{C}}(T \cdot x) = 1\}$ is a sphere (see [24], [10]). The closure of $\{x \in \mathcal{H}_\lambda : \dim_{\mathbb{C}}(T \cdot x) = 1\}$ is called 1-skeleton of \mathcal{H}_λ . The **moment graph** or **GKM graph** of \mathcal{H}_λ is the image of its 1-skeleton under the momentum map. This graph has vertices corresponding to the T -fixed points and edges corresponding to closures of connected components of the 1-skeleton. Two vertices are connected by an edge in the moment graph if and only if they differ by one transposition.

For two T -fixed points $F, F' \in \mathcal{H}_\lambda$ such that their images under the momentum map μ are connected by an edge in the moment graph, we denote by $S_{F, F'}^2 \subset \mathcal{H}_\lambda$ the corresponding sphere associated to them.

We now want to compute the symplectic area of $S_{F, F'}^2 \subset \mathcal{H}_\lambda$ with respect to ω_λ in terms of λ . Let us suppose that F and F' differ by the transposition $(i, j) \in S_n$ and the i -th component $F_i \in \{\lambda_1, \dots, \lambda_n\}$ of F is greater than its j -th component $F_j \in \{\lambda_1, \dots, \lambda_n\}$. If $T' \subset T$ is the codimension one torus that fixes $S_{F, F'}^2$, there exists a torus of dimension one $S \subset T$ such that $T \cong T' \times S$. We will use the identification $S := \mathbb{R}/\mathbb{Z}$, which induces an isomorphism $\text{Lie}(S) \cong \mathbb{R}$ leading to $\text{Lie}(S)^* \cong \mathbb{R}$, mapping the lattice $\text{Hom}(S, S^1) \subset \text{Lie}(S)^*$ isomorphically to $\mathbb{Z} \subset \mathbb{R}$.

The action of S on $S_{F,F'}^2$ is hamiltonian with momentum map

$$\iota^* \circ \mu|_{S_{F,F'}^2} : S_{F,F'}^2 \rightarrow \text{Lie}(S)^* \cong \mathbb{R},$$

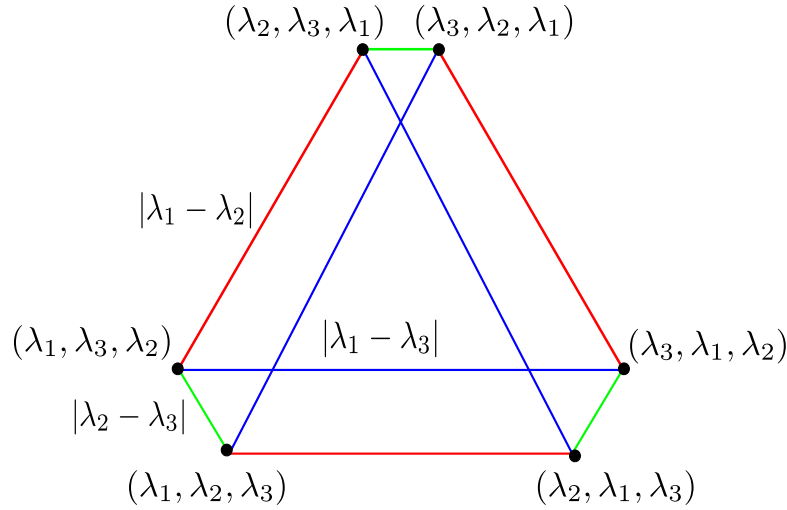
where $\iota : S \hookrightarrow T$ is the inclusion map. The momentum image of $S_{F,F'}^2$ under $\iota^* \circ \mu|_{S_{F,F'}^2}$ is the segment line that joins $\iota^*(\mu(F))$ with $\iota^*(\mu(F'))$. Note that the weight of T on $T_F S_{F,F'}^2$ is equal to $e_i - e_j$, thus the weight of the action of S on $T_F S_{F,F'}^2$ is $\iota^*(e_i - e_j)$.

Let $\gamma : [0, 1] \rightarrow S_{F,F'}^2 \hookrightarrow \mathcal{H}_\lambda$ be any smooth path from F to F' and $c : [0, 1] \times S \rightarrow S_{F,F'}^2$ be the map defined by $c(t, s) := s \cdot \gamma(t)$. Then,

$$\int_{[0,1] \times S} c^*(\omega_\lambda|_{S_{F,F'}^2}) = \int_0^1 \gamma^*(\iota_{\xi_{S_{F,F'}^2}} \omega_\lambda) = \iota^*(\mu(F)) - \iota^*(\mu(F')).$$

Note that the integral $\int_{[0,1] \times S} c^* \omega_\lambda$ is equal to the symplectic area of $S_{F,F'}^2$ times the weight $\iota^*(e_i - e_j)$. Since $F - F' = (F_i - F_j)(e_i - e_j)$, and $\iota^*(\mu(F)) - \iota^*(\mu(F')) = (F_i - F_j)\iota^*(e_i - e_j)$, we conclude that the symplectic area of $S_{F,F'}^2$ is equal to $F_i - F_j$.

As an example, the following figure shows the moment graph of $\mathcal{H}_{(\lambda_1, \lambda_2, \lambda_3)}$ with three of its edges labeled with their corresponding symplectic areas:



Let us suppose now that $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ is of the form

$$\lambda_1 = \dots = \lambda_{m_1}, \lambda_{m_1+1} = \dots = \lambda_{m_1+m_2}, \dots, \lambda_{m_1+m_2+\dots+m_{l-1}+1} = \dots = \lambda_n,$$

where $1 \leq m_1, m_2, \dots, m_{l-1}, m_l \leq n$ are integers such that $m_1 + m_2 + \dots + m_{l-1} + m_l = n$, and $\{\lambda_{m_1}, \lambda_{m_1+m_2}, \dots, \lambda_n\}$ are all the pairwise different components of λ . Let a be the strictly increasing sequence of integers $0 = a_0 < a_1 < a_2 < \dots < a_l = n$ defined by $a_j = \sum_{i=1}^j m_i$ and let $Fl(a; n)$ be the set of flags of type a , i.e., the set of increasing filtrations of \mathbb{C}^n by complex subspaces

$$0 = V^0 \subset V^1 \subset V^2 \subset \dots \subset V^l = \mathbb{C}^n$$

such that $\dim_{\mathbb{C}} V^i = a_i$. Note that there is a naturally defined action of $Sl(n, \mathbb{C})$ on $Fl(a; n)$.

For a flag $V = (V^1, \dots, V^l) \in Fl(a; n)$, denote by $P_j = P_j(V)$ the orthogonal projection onto V_j . We can form the Hermitian operator

$$A_\lambda(V) = \sum_j \lambda_{a_j} (P_j - P_{j-1}).$$

The correspondence $V \mapsto A_\lambda(V)$ defines a diffeomorphism between $Fl(a; n)$ and \mathcal{H}_λ . This diffeomorphism defines by pullback a $U(n)$ -invariant symplectic form on $Fl(a; n)$. It also defines an integrable almost complex structure on $Fl(a; n)$ so that $Sl(n, \mathbb{C})$ acts holomorphically on $Fl(n, \mathbb{C})$, and the map $A_\lambda : Fl(n, \mathbb{C}) \rightarrow \mathcal{H}_\lambda$ is a $Sl(n, \mathbb{C})$ -invariant biholomorphism.

The (co)homology of $Fl(a, n)$ (and hence the (co)homology of \mathcal{H}_λ) can be computed from the CW-structure of $Fl(a; n)$ coming from its Schubert cell decomposition.

Let S_n be the group of permutations of n elements. Recall that the length of a permutation is, by definition, equal to the smallest number of adjacent transpositions whose product is the permutation. Let $W_a \subset S_n$ be the subgroup generated by the simple transpositions $s_i = (i, i+1)$ for $i \notin \{a_1, \dots, a_l\}$. Let $W^a \subset S_n$ be the set of smallest coset representatives of S_n/W_a . Let $F \in Fl(a; n)$ be the partial flag defined by

$$F := \mathbb{C}^{a_1} \subset \mathbb{C}^{a_2} \subset \dots \subset \mathbb{C}^{a_n} = \mathbb{C}^n$$

and B be the standard Borel subgroup of $Sl(n, \mathbb{C})$ of upper triangular matrices.

For a permutation $w \in W^a$, the **Schubert cell** C_w is the orbit of the induced action of $B \subset Sl(n, \mathbb{C})$ on $Fl(a; n)$ through $w \cdot F$. The **Schubert variety** X_w is by definition the closure of the Schubert cell C_w .

For $w \in W^a$, the Schubert cell C_w is isomorphic to an affine space of complex dimension equal to the length of w . The Schubert cells $\{C_w\}_{w \in W^a}$ define a CW-complex for $Fl(a; n)$ with cells occurring only in even dimension. Thus, the fundamental classes $[X_w]$ of $X_w, w \in W^a$, are a free basis of $H_*(Fl(a; n), \mathbb{Z})$ as a \mathbb{Z} -module. Likewise, the Poincaré dual classes of $[X_w]$, $w \in W^a$, are a free basis of $H^*(Fl(a; n), \mathbb{Z})$ as a \mathbb{Z} -module.

The diffeomorphism $A_\lambda : Fl(a; n) \rightarrow \mathcal{H}_\lambda$ maps the Schubert cells $C_w \in Fl(a; n), w \in W^a$, to the B -orbits of $w \cdot \lambda$ in \mathcal{H}_λ . By abusing notation, we will denote the B -orbits of $w \cdot \lambda$ in \mathcal{H}_λ by C_w and their closures by X_w and refer to them as the Schubert cells and Schubert varieties associated to $w \in W^a$ in \mathcal{H}_λ , respectively.

Remark 3.1. Note that A_λ maps the Schubert varieties $X_{(a_j, a_{j+1})} \subset Fl(a; n)$ to the spheres $S^2_{\lambda, (a_j, a_{j+1}) \cdot \lambda} \subset \mathcal{H}_\lambda$. Thus, the homology group $H_2(\mathcal{H}_\lambda, \mathbb{Z})$ is freely generated as a \mathbb{Z} -module by the fundamental classes of $S^2_{\lambda, (a_j, a_{j+1}) \cdot \lambda}, 1 \leq j \leq l$.

4. UPPER BOUNDS OF THE GROMOV WIDTH OF GRASSMANNIAN MANIFOLDS

Yael Karshon and Susan Tolman in [15] found upper bounds for the Gromov width of Grassmannian manifolds by computing a Gromov-Witten invariant. In this section, we are going to review this idea, which would be particularly useful for considering the most general problem of determining upper bounds for the Gromov width of partial flag manifolds.

We establish the convention that would be used during this section. Let $G(k, n)$ be the Grassmannian manifold of k -planes in \mathbb{C}^n . Let $\lambda \in \mathbb{R}^n$ be of the form

$$\lambda_1 = \dots = \lambda_1 > \lambda_2 = \dots = \dots \lambda_2$$

and $\mathcal{H}_\lambda = \{A \in M_n(\mathbb{C}) : A^* = A, \text{ spectrum } A = \lambda\}$. As we have remarked in the previous section, there is some integer $1 \leq k \leq n$ such that \mathcal{H}_λ is diffeomorphic to a Grassmannian manifold $G(k, n)$.

Let $(\omega_\lambda, J_\lambda)$ be the Kähler structure of $\mathcal{H}_\lambda \cong G(k, n)$ defined in Section 3. Let A be the standard generator of the second homology group $H_2(G(k, n), \mathbb{Z})$. Let

$$\mathcal{M}_A(J_\lambda) = \{u : \mathbb{CP}^1 \rightarrow G(k, n) : u \text{ is } J_\lambda\text{-holomorphic and } u_*[\mathbb{CP}^1] = A\}$$

be the moduli space of J_λ -holomorphic curves of degree A defined on $G(k, n)$. This moduli space is usually called the space of projective lines of the Grassmannian manifold $G(k, n)$.

For a holomorphic curve $u : \mathbb{CP}^1 \rightarrow G(k, n)$ of degree A , we define the kernel of u as the intersection of all the subspaces $V \subset \mathbb{C}^n$ that are in the image of u . Similarly, the span of u is the linear span of these subspaces:

$$\ker(u) = \bigcap_{V \in u(\mathbb{CP}^1)} V, \quad \text{span}(u) = \sum_{V \in u(\mathbb{CP}^1)} V.$$

The kernel and span of u are of dimension $k - 1$ and $k + 1$ respectively and they determine uniquely, up to parametrization, the holomorphic curve u , i.e., if there is a holomorphic curve $v : \mathbb{CP}^1 \rightarrow G(k, n)$ of degree A such that $\ker(u) = \ker(v)$ and $\text{span}(u) = \text{span}(v)$, then there exists $g : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \in PSL(2; \mathbb{C})$ such that $v = g \circ u$. Moreover, $u(\mathbb{CP}^1) = \{V^k \in G(k, n) : \ker(u) \subset V^k \subset \text{span}(u)\} \subset G(k, n)$ [3]. So $\mathcal{M}_A(J_\lambda)/PSL(2, \mathbb{C}) \simeq Fl(k - 1, k + 1; n)$, where $Fl(k - 1, k + 1; n)$ denotes the partial flag manifold of complex subspaces sequences

$$V^{k-1} \subset V^{k+1} \subset \mathbb{C}^n.$$

For $V = (V^{k-1}, V^{k+1}) \in Fl(k - 1, k + 1; n)$, we will denote by u_V the projective line

$$\mathbb{CP}^1 \simeq u_V = \{V^k \in G(k, n) : V^{k-1} \subset V^k \subset V^{k+1}\} \subset G(k, n).$$

Notice that $\mathcal{M}_A(J_\lambda)/PSL(2, \mathbb{C})$ is compact due to the indecomposability of A . Let us consider the evaluation map

$$\text{ev}_{J_\lambda}^2 : \mathcal{M}_A(J_\lambda) \times_{PSL(2, \mathbb{C})} (\mathbb{CP}^1)^2 \rightarrow G(k, n)^2.$$

We want to find a compact complex submanifold $X \subset G(k, n)$ such that for a generic point p in $G(k, n)$ the evaluation map $\text{ev}_{J_\lambda}^2$ would be transverse to $(\{p\} \times X) \subset G(k, n)^2$, $\dim_{\mathbb{C}}(\mathcal{M}_A(J_\lambda) \times_{PSL(2, \mathbb{C})} (\mathbb{CP}^1)^2) + \dim_{\mathbb{C}} X$ would be equal to $2 \dim_{\mathbb{C}} G(k, n)$, and the number of holomorphic curves in $\mathcal{M}_A(J_\lambda)/PSL(2, \mathbb{C})$ that pass through p and X would be different from zero. If so, the Gromov-Witten invariant $\text{GW}_{A, 2}^{J_\lambda}(\text{PD}[p], \text{PD}[X])$ would be different from zero and by Theorem 2.3 and Remark 2.4, we will have that

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq \omega_\lambda(A) = |\lambda_1 - \lambda_2|.$$

We claim that the Grassmannian manifold $X = \{V^k \in G(k, n) : \mathbb{C} \subset V^k \subset \mathbb{C}^{n-1}\} \subset G(k, n)$ satisfies all these conditions.

Proving that the evaluation map $\text{ev}_{J_\lambda}^2$ is transverse to $(\{p\} \times X) \subset G(k, n)^2$ can be obtained as a consequence of the Bertini-Kleiman Transversality Theorem:

Theorem 4.1. Bertini, Kleiman [20] *Let $f : U \rightarrow V$ be a smooth map between smooth manifolds and let G be a Lie group that acts transitively on V . Let Z be an arbitrary submanifold of V and G^{reg} be the set of elements $g \in G$ for which f is transverse to gZ . Then, G^{reg} is a set of the second category in G .*

We now prove that indeed the Gromov-Witten invariant $\text{GW}_{A,2}^{J_\lambda}(\text{PD}[p], \text{PD}[X])$ is different from zero.

Lemma 4.2. *Let $X = \{V^k \in G(k, n) : \mathbb{C} \subset V^k \subset \mathbb{C}^{n-1}\} \simeq G(k-1, n-2)$ and $p \in G(k, n)$. Then*

$$\text{GW}_{A,2}^{J_\lambda}(\text{PD}[p], \text{PD}[X]) = 1.$$

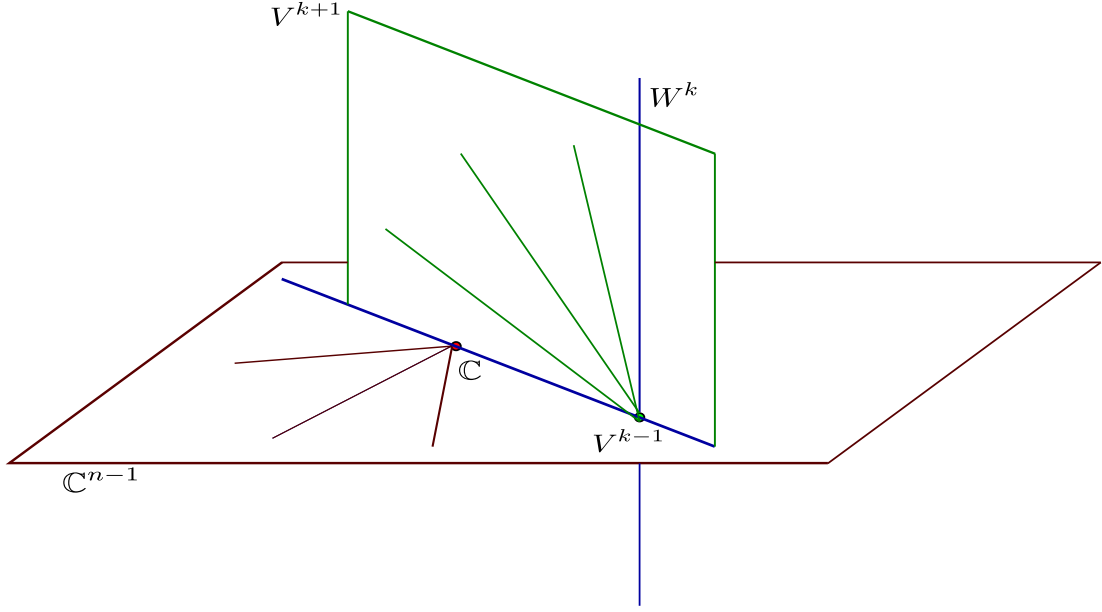
Proof. Since the complex dimension of X is equal to $(n-k-1)(k-1)$, X satisfies the dimensional constraint

$$\begin{aligned} \dim_{\mathbb{C}}(\mathcal{M}_A(J_\lambda) \times_{PSL(2, \mathbb{C})} (\mathbb{CP}^1)^2) + \dim_{\mathbb{C}} X &= \dim_{\mathbb{C}} Fl(k-1, k+1; n) + 2 + \dim_{\mathbb{C}} X \\ &= 2 \dim_{\mathbb{C}} G(k, n). \end{aligned}$$

Assume now that $p = W^k$ is a k -dimensional subspace of \mathbb{C}^n that does not contain \mathbb{C} and transversally intersects \mathbb{C}^{n-1} . We claim that $(\text{ev}_{J_\lambda}^2)^{-1}(\{p\} \times X)$ consists of just one element, i.e., there is a unique line in $G(k, n)$ that intersects X and passes through W^k : let $V = (V^{k-1}, V^{k+1}) \in Fl(k-1, k+1; n)$ such that the projective line u_V passes through both X and p . So there exists $V^k \in X$ (that is, $\mathbb{C} \subset V^k \subset \mathbb{C}^{n-1}$) and $V^{k-1} \subset V^k \subset V^{k+1}$. Moreover we have $V^{k-1} \subset W^k \subset V^{k+1}$ (W^k is p).

Note that, we have inclusions $V^{k-1} \subset \mathbb{C}^{n-1}$ and $V^{k-1} \subset W^k$. Thus $V^{k-1} \subset W^k \cap \mathbb{C}^{n-1}$. But $W^k \cap \mathbb{C}^{n-1}$ is a $(k-1)$ -dimensional vector subspace because the intersection is transverse. Thus $V^{k-1} = W^k \cap \mathbb{C}^{n-1}$. The intersection $V^{k-1} = W^k \cap \mathbb{C}^{n-1}$ does not contain \mathbb{C} . So there exists a unique k -dimensional vector space U^k such that $V^{k-1} \subset U^k$ and $\mathbb{C} \subset U^k \subset \mathbb{C}^{n-1}$. This vector space is $U^k = V^{k-1} \oplus \mathbb{C}$. Thus, $V^k = V^{k-1} \oplus \mathbb{C}$. The vector space V^{k+1} contains W^k and $V^k = V^{k-1} \oplus \mathbb{C}$. Observe that V^k is different from W^k because V^k contains \mathbb{C} and W^k does not. Therefore $V^{k+1} = W^k + V^k$.

In conclusion $(V^{k-1}, V^{k+1}) = (W^k \cap \mathbb{C}^{n-1}, W^k + ((W^k \cap \mathbb{C}^{n-1}) \oplus \mathbb{C}))$, which determines a unique projective line that intersects X and passes through W^k .



Note that if $p = W^k$ is a k -dimensional subspace of \mathbb{C}^n that either contains \mathbb{C} or is contained in \mathbb{C}^{n-1} , then $(\text{ev}_{J_\lambda}^2)^{-1}(\{p\} \times X)$ consists of an infinite number of elements.

We now prove that the evaluation map

$$\text{ev}_{J_\lambda}^2 : \mathcal{M}_{A,2}(J_\lambda) \rightarrow G(k, n)^2$$

is transverse to $(\{p\} \times X) \subset G(k, n)^2$. The group $Sl(n, \mathbb{C})$ acts transitively and holomorphically on $G(k, n)$ so as a consequence there exists $h \in Sl(n, \mathbb{C})$ such that $\text{ev}_{J_\lambda}^2 \pitchfork (\{h \cdot p\} \times X) \subset G(k, n)^2$ and thus the preimage $(\text{ev}_{J_\lambda}^2)^{-1}(\{h \cdot p\} \times X)$ consists of just one point (the number of elements of the preimage $(\text{ev}_{J_\lambda}^2)^{-1}(\{h \cdot p\} \times X)$ is either one or infinite, but if the evaluation map is transverse to $\{h \cdot p\} \times X$ it has to be necessarily one), by Proposition 7.4.5 of [20] the Gromov-Witten invariant $\text{GW}_{A,2}^{J_\lambda}(\text{PD}[p], \text{PD}[X])$ is positive, so in conclusion

$$\text{GW}_{A,2}^{J_\lambda}(\text{PD}[p], \text{PD}[X]) = \text{GW}_{A,2}^{J_\lambda}(\text{PD}[h \cdot p], \text{PD}[X]) = 1$$

□

We have proved that for Grassmannian manifolds there is a non-vanishing Gromov-Witten invariant with one of its constrains being a point. This would imply that the Gromov width of a Grassmannian manifolds is bounded from above by the symplectic area of any line of the Grassmannian manifold. In summary, we have the following result:

Theorem 4.3 (Karshon-Tolman, Guangcun Lu). *Let*

$$\mathcal{H}_\lambda = \{A \in M_n(\mathbb{C}) : A^* = A, \text{spectrum } A = \lambda\}$$

where $\lambda \in \mathbb{R}^n$ is of the form

$$\lambda_1 = \cdots = \lambda_1 > \lambda_2 = \cdots = \cdots \lambda_2,$$

and let ω_λ be the Kirillov-Kostant-Souriau form defined on \mathcal{H}_λ . Then,

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq |\lambda_1 - \lambda_2|.$$

Proof. The result follows from Remark 2.4 and Lemma 4.2, and the fact that the symplectic area of A with respect to ω_λ is equal to $|\lambda_1 - \lambda_2|$. \square

5. UPPER BOUNDS OF THE GROMOV WIDTH OF COADJOINT ORBITS OF TYPE A

The problem of finding upper bounds of the Gromov width of coadjoint orbits of type A has already been addressed by Masrour Zoghi in his Ph.D thesis [25] where he has considered the problem of determining the Gromov width of regular coadjoint orbits of compact Lie groups. We start this section by first describing Zoghi's results, and then we show how to extend his results to coadjoint orbits that may no be regular.

Theorem 5.1. *Let (M, ω) be a symplectic manifold and J be a regular ω -compatible almost complex structure on M , and suppose that M admits a J -holomorphic \mathbb{CP}^1 -fibration $\pi : M \rightarrow Y$ where Y is a connected, compact Kähler manifold, and let $d \in H_2(M, \mathbb{Z})$ denote the homology class of the fibers of π . Then, the evaluation map*

$$\text{ev}_J^1 : \mathcal{M}_{d,1}^*(M, J) \rightarrow M$$

is a diffeomorphism.

Sketch: Let $u : \mathbb{CP}^1 \rightarrow M$ be a holomorphic curve of degree d . Note that $\pi_* u_* [\mathbb{CP}^1] = \pi_*(d) = 0$. This implies that the map $\pi \circ u$ is constant because Y is a connected, compact Kähler manifold. As a consequence, the image of u lies totally in a fiber of $\pi : M \rightarrow Y$, let's say $F \cong \mathbb{CP}^1$. The map $u : \mathbb{CP}^1 \rightarrow F \cong \mathbb{CP}^1$ is holomorphic of degree one; and thus $u : \mathbb{CP}^1 \rightarrow F$ is a biholomorphism. In conclusion, the J -holomorphic curves of M of degree d , up to parametrization, are embedded curves in M and correspond to the fibers of $\pi : M \rightarrow Y$.

We claim (but we will no prove it) that the map $\rho : \mathcal{M}_{d,0}^*(M, J) \rightarrow Y$ that sends one J -holomophirc map $u : \mathbb{CP}^1 \rightarrow M$ to the point $\pi \circ u(\mathbb{CP}^1) \in Y$ is indeed a diffeomorphism.

Now, if $f : \mathcal{M}_{d,1}^*(M, J) \rightarrow \mathcal{M}_{d,0}^*(M, J)$ denotes the forgetful map, the following diagram

$$\begin{array}{ccc} \mathcal{M}_{d,1}^*(M, J) & \xrightarrow{\text{ev}_J^1} & M \\ \downarrow f & & \downarrow \pi \\ \mathcal{M}_{d,0}^*(M, J) & \xrightarrow{\rho} & Y \end{array}$$

is commutative. It is not difficult to see that the fibers of f are mapped diffeomorphically onto the fibers of π . This together with the fact that $\rho : \mathcal{M}_{d,0}^*(M, J) \rightarrow Y$ is a diffeomorphism implies that ev_J^1 is a diffeomorphism. \square

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, $\mathcal{H}_\lambda = \{A \in M_n(\mathbb{C}) : A^* = A, \text{spectrum } A = \lambda\}$ and $(\omega_\lambda, J_\lambda)$ be the Kähler structure of \mathcal{H}_λ defined in Section 3.

The following theorem appears in Zoghi's Ph.D thesis [25] as one of its main results:

Theorem 5.2 (Zoghi). *Let $\lambda \in \mathbb{R}^n$ be of the form $\lambda_1 > \cdots > \lambda_n$. Suppose that there is an integer k such that any difference of eigenvalues $\lambda_i - \lambda_j$ is an integer multiple of $\lambda_{k+1} - \lambda_k$, then*

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq |\lambda_k - \lambda_{k+1}|.$$

Proof. The flag variety $Fl(n)$ of sequences of complex vector spaces

$$V^1 \subset V^2 \subset \cdots \subset V^n = \mathbb{C}^n$$

is isomorphic to \mathcal{H}_λ . The second homology group $H_2(\mathcal{H}_\lambda, \mathbb{Z})$ is freely generated by fundamental classes of Schubert varieties $X_{(j,j+1)}$ parameterized by the transpositions $(j, j+1) \in S_n$. By assumption, there exists $1 \leq k < n$ such that the symplectic areas $\omega_\lambda(X_{(i,i+1)}) = |\lambda_i - \lambda_{i+1}|$ are integer multiples of the symplectic area $\omega_\lambda(X_{(k,k+1)}) = |\lambda_k - \lambda_{k+1}|$ for $1 \leq i < n$. This implies that $[X_{(k,k+1)}]$ is a ω_λ -indecomposable homology class.

We have a naturally defined holomorphic fibration

$$\pi_k : Fl(n) \rightarrow Fl(1, \dots, \widehat{k}, \dots, n-1; n)$$

with fiber isomorphic to \mathbb{CP}^1 . Note that the fundamental class $[X_{(k,k+1)}]$ is the homology class of the fiber of π_k .

By Theorem 5.1, the evaluation map

$$ev_{J_\lambda}^1 : \mathcal{M}_{[X_{(k,k+1)}], 1}^*(\mathcal{H}_\lambda, J_\lambda) \rightarrow \mathcal{H}_\lambda$$

is a diffeomorphism; in particular, it has degree one. Since $[X_{(k,k+1)}]$ is a ω_λ -indecomposable homology class, for regular ω_λ -compatible almost complex structures J' , the moduli spaces of J' -holomorphic maps $\mathcal{M}_{[X_{(k,k+1)}], 1}^*(\mathcal{H}_\lambda, J')$ are compact and the evaluation maps $ev_{J'}^1$ are compactly cobordant among each other. In particular, for regular ω_λ -compatible almost complex structures J' , the evaluation maps

$$ev_{J'}^1 : \mathcal{M}_{[X_{(k,k+1)}], 1}^*(\mathcal{H}_\lambda, J') \rightarrow \mathcal{H}_\lambda$$

have degree one and hence they are onto, which by Theorem 2.3 implies that

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq \omega_\lambda[X_{(k,k+1)}] = |\lambda_k - \lambda_{k+1}|.$$

□

We now prove the main result of this paper, which extends Zoghi's result to coadjoint orbits that are not necessarily regular. But first we state the following lemma:

Lemma 5.3. *Let $G = Sl(n, \mathbb{C})$, B be the subgroup of G consisting of upper triangular matrices and $P \subset B$ be a parabolic subgroup of block upper triangular matrices. Let X be an algebraic G -variety and $\pi : X \rightarrow G/P$ be an equivariant map. If \check{X} is the B -stable open dense Schubert cell of G/P , then π is a trivial fibration over \check{X} .*

Proof. Let $x_0 \in \mathring{X}$ be any point and $U \subset B$ be the unipotent radical of P . The map $s : U \rightarrow \mathring{X}$ defined by $g \mapsto g \cdot x_0$ is an isomorphism. So that, the map

$$\begin{aligned} \psi : \mathring{X} \times \pi^{-1}(x_0) &\rightarrow \pi^{-1}(\mathring{X}) \\ (x, y) &\mapsto s(x) \cdot y \end{aligned}$$

is an isomorphism with inverse given by $\psi^{-1}(m) = (\pi(m), s(\pi(m))^{-1} \cdot m)$. \square

Theorem 5.4. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Suppose that there are i, j such that any difference of eigenvalues $\lambda_{i'} - \lambda_{j'}$ is an integer multiple of $\lambda_i - \lambda_j$, then*

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq |\lambda_i - \lambda_j|$$

Proof. The idea of the proof is, as before, to prove that a certain Gromov-Witten invariant, with one of its constraints being a point, is different from zero.

Let us assume that $\lambda \in \mathbb{R}^n$ is of the form

$$\lambda_1 = \dots = \lambda_{m_1}, \lambda_{m_1+1} = \dots = \lambda_{m_1+m_2}, \dots, \lambda_{m_1+m_2+\dots+m_{l-1}+1} = \dots = \lambda_n,$$

where $1 \leq m_1, m_2, \dots, m_{l-1}, m_l \leq n$ are integers such that $m_1 + m_2 + \dots + m_{l-1} + m_l = n$, and $\lambda_{m_1}, \lambda_{m_1+m_2}, \dots, \lambda_n$ are pairwise different real numbers. After reordering the components of λ if necessary, we assume that $i = m_1 + 1, j = m_1$ so that $\lambda_{m_1+1} - \lambda_{m_1}$ is an integer multiple of any difference of the form $\lambda_{i'} - \lambda_{j'}$.

We know that $\mathcal{H}_\lambda \simeq Fl(a; n)$, where a is the strictly increasing sequence of integers

$$0 = a_0 < a_1 < \dots < a_l = n$$

defined by $a_k = \sum_{r=1}^k m_r$, for $1 \leq k \leq l$.

Let a' be the sequence of integer numbers

$$a_2 < \dots < a_l = n,$$

and $Fl(a'; n)$ be the corresponding flag manifold. Let $W_a \subset S_n$ be the subgroup generated by the simple transpositions $s_i = (i, i+1)$ for $i \notin \{a_1, \dots, a_l\}$. Let $W^a \subset S_n$ be the set of smallest coset representatives of S_n/W_a . Likewise, we define $W_{a'}$ and $W^{a'}$. Schubert varieties of $Fl(a; n)$ and $Fl(a'; n)$ are parametrized by W^a and $W^{a'}$, respectively. To avoid confusions, we will denote the Schubert varieties in $Fl(a; n)$ by X_\bullet and the Schubert varieties in $Fl(a'; n)$ by X'_\bullet . A similar thing will be done with the Schubert cells.

For the permutations $(a_1, a_1 + 1) \in W^a$, let $X_{(a_1, a_1+1)}$ be the standard Schubert variety in $Fl(a; n)$ associated to it and let A be the the fundamental class of this Schubert variety. Note that, by assumption, $\omega_\lambda(A) = |\lambda_{m_1+1} - \lambda_{m_1}|$ is a generator of the cyclic image $\omega_\lambda(H_2(\mathcal{H}_\lambda, \mathbb{Z}))$, which implies that A is a ω_λ -indecomposable homology class. As a consequence, the Gromov-Witten invariant $\text{GW}_{A,k}$ is well defined.

We have a holomorphic projection

$$\begin{aligned} \pi : Fl(a; n) &\rightarrow Fl(a'; n) \\ V^{a_1} \subset V^{a_2} \subset \dots \subset V^{a_l} = \mathbb{C}^n &\mapsto V^{a_2} \subset \dots \subset V^{a_l} = \mathbb{C}^n \end{aligned}$$

whose fiber is isomorphic to a Grassmanian manifold $G(a_1, a_2)$. If $G(a_1, a_2)$ is isomorphic to \mathbb{CP}^1 , we are in the case of Theorem 5.1, and we are done.

The set of minimal length representatives $W_a^{a'}$ of $W_{a'}$ on W_a parameterizes Schubert varieties on a fiber of π . Note that $(a_1, a_{1+1}) \in W_a^{a'}$, so in particular $\pi_*(A) = 0$.

Let \tilde{w} be the permutation in $W_a^{a'}$ that represents in a fiber a Grassmannian manifold isomorphic to $G(a_1 - 1, a_2 - 2)$. Let w' be the longest element in $W^{a'}$. The Schubert cell $C'_{w'}$ is open and dense in $Fl(a'; n)$. By the previous Lemma, the restriction map

$$\pi|_{X_{w'\tilde{w}}} : X_{w'\tilde{w}} \rightarrow Fl(a, n)$$

is a trivial fibration over $C'_{w'}$ with fiber isomorphic to $G(a_1 - 1, a_2 - 2)$.

We now want to count the number of holomorphic curves of degree A that passes through a generic point $p \in Fl(a; n)$ and $X_{w'\tilde{w}} \subset Fl(a; n)$. Let $u : \mathbb{CP}^1 \rightarrow Fl(a; n)$ be one of such holomorphic curves. The composition $\pi \circ u$ is holomorphic and $(\pi \circ u)_*[\mathbb{CP}^1] = \pi_*(A) = 0$. Since $Fl(a'; n)$ is a compact and connected Kähler manifold, the map $\pi \circ u$ is constant, which means that the image of $u : \mathbb{CP}^1 \rightarrow Fl(a; n)$ lies entirely in the fiber $\pi^{-1}(p) \cong G(a_1, a_2)$ of $\pi : Fl(a; n) \rightarrow Fl(a'; n)$. Moreover, $u : \mathbb{CP}^1 \rightarrow \pi^{-1}(p) \cong G(a_1, a_2) \subset Fl(a; n)$ is a holomorphic map of degree one, i.e., it is a projective line of the fiber $\pi^{-1}(p) \cong G(a_1, a_2)$. If $\pi(p) \in C'_{w'}$, then the fiber $\pi^{-1}(p)$ intersects $X_{w'\tilde{w}}$ in a variety isomorphic to $G(a_1 - 1, a_2 - 2)$. Since there is just one projective line passing through a generic point and $G(a_1 - 1, a_2 - 2)$ in $G(a_1, a_2)$ (by Lemma 4.2), we conclude that

$$\text{GW}_{A,2}^{J_\lambda}(\text{PD}[p], \text{PD}[X_{w'\tilde{w}}]) = 1.$$

Thus, by Theorem 2.3 and Remark 2.4,

$$\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq \omega_\lambda(A) = |\lambda_{m_1+1} - \lambda_{m_1}|.$$

□

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